



University of Pennsylvania  
**ScholarlyCommons**

---

Publicly Accessible Penn Dissertations

---

1-1-2015

# Twisted Spectral Data and Singular Monopoles

Tong Li

University of Pennsylvania, [litong@math.upenn.edu](mailto:litong@math.upenn.edu)

Follow this and additional works at: <http://repository.upenn.edu/edissertations>

 Part of the [Mathematics Commons](#)

---

## Recommended Citation

Li, Tong, "Twisted Spectral Data and Singular Monopoles" (2015). *Publicly Accessible Penn Dissertations*. 1089.  
<http://repository.upenn.edu/edissertations/1089>

This paper is posted at ScholarlyCommons. <http://repository.upenn.edu/edissertations/1089>

For more information, please contact [libraryrepository@pobox.upenn.edu](mailto:libraryrepository@pobox.upenn.edu).

---

# Twisted Spectral Data and Singular Monopoles

**Abstract**

We study higher dimensional versions of monopoles with Dirac singularities on manifolds which are principal circle bundles over a smooth complex projective variety. We interpret such generalized monopoles in terms of twisted spectral data on a companion algebraic variety. We conjecture that this correspondence is bijective under certain stability condition, and thus gives an algebraic construction of singular monopoles.

**Degree Type**

Dissertation

**Degree Name**

Doctor of Philosophy (PhD)

**Graduate Group**

Mathematics

**First Advisor**

Tony Pantev

**Keywords**

Dirac singularity, Fourier-Mukai transform, gerbe, Kobayashi-Hitchin correspondence, monopole, twisted spectral data

**Subject Categories**

Mathematics

# TWISTED SPECTRAL DATA AND SINGULAR MONOPOLES

Tong Li

A DISSERTATION

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial  
Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2015

Supervisor of Dissertation

---

Tony Pantev, Professor of Mathematics

Graduate Group Chairperson

---

David Harbater, Christopher H. Browne Distinguished Professor in  
the School of Arts and Sciences

Dissertation Committee:

Tony Pantev, Professor of Mathematics

Ron Donagi, Professor of Mathematics

Jonathan Block, Professor of Mathematics

# Acknowledgments

I owe my deepest gratitude to my advisor, Tony Pantev, for suggesting the problem to me and for his constant support and tremendous help throughout my graduate study. He has always been eager to discuss mathematics and share with me his acute intuition and vast knowledge.

I also thank Ron Donagi and Jonathan Block for teaching me geometry, for being on my committee and for valuable remarks concerning my thesis.

I would like to thank several others who enriched my mathematical life. I specially thank Alberto García-Raboso, Ryan Manion, Ana Peón-Nieto, Tyler Kelly, Ted Spaide, Zhentao Lu, Ying Zhang, Zhaoting Wei, Taisong Jing, Haomin Wen, Shilin Yu, Xiaofei Jin, Antonijo Mrcela and Jin Woo Jang for making my experience here at Penn so enjoyable. I thank Janet Burns, Monica Pallanti, Paula Scarborough and Reshma Tanna for making the department like a family to me.

Last but not least, I am grateful to my father Baoxing Li, my mother Xiuyun Li and my adorable wife Yu Wang for their encouragements and persistent love.

# ABSTRACT

## TWISTED SPECTRAL DATA AND SINGULAR MONOPOLES

Tong Li

Tony Pantev

We study higher dimensional versions of monopoles with Dirac singularities on manifolds which are principal circle bundles over a smooth complex projective variety. We interpret such generalized monopoles in terms of twisted spectral data on a companion algebraic variety. We conjecture that this correspondence is bijective under certain stability condition, and thus gives an algebraic construction of singular monopoles.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Singular monopole</b>	<b>4</b>
2.1	Monopole equation . . . . .	4
2.2	Dirac singularity . . . . .	8
<b>3</b>	<b>Charbonneau-Hurtubise theorem</b>	<b>11</b>
3.1	The scattering map . . . . .	11
3.2	Stability . . . . .	14
<b>4</b>	<b>Spectral data</b>	<b>16</b>
4.1	Spectral datum of a bundle pair . . . . .	16
4.2	Fourier-Mukai transform . . . . .	18
4.3	Analytic gerbe . . . . .	21
4.4	Twisted spectral data . . . . .	24
<b>5</b>	<b>Conjecture</b>	<b>30</b>

# Chapter 1

## Introduction

A monopole is a hypothetical elementary particle that is an isolated magnet with only one magnetic pole. Dirac described in [Dir31] a family of singular monopoles on  $\mathbb{R}^3$ , and he showed that the existence of monopole leads to a natural quantization of electric charges. Since then, there have been extensive studies about monopole in both physics and mathematics.

In gauge theory, one describes a monopole as a Yang-Mills potential  $A$  on a Hermitian vector bundle  $E$  and Higgs field  $\phi \in \text{End}(E)$  whose equation of motion is given by the action:

$$\int (F_A, F_A) + (d_A \phi, d_A \phi) + \lambda(1 - \|\phi\|^2)^2.$$

In mathematics, one often consider static solutions by setting  $\lambda = 0$ , and refer monopole to a static solution instead of a dynamic one. The equation of motion for

a static monopole is the Bogomolny's equation:

$$F_A = *d_A \phi.$$

Bogomolny's equation has holomorphic interpretations, which enables one to understand monopoles using tools from algebraic geometry. For example, Hitchin in [Hit82] studied  $SU(2)$  monopoles on  $\mathbb{R}^3$  using twistor methods, he showed that every such monopole can be constructed canonically from an algebraic curve.

On compact spaces, global smooth monopoles are less interesting, and people often attach Dirac singularity to it. In [CH11], Charbonneau and Hurtubise studied singular monopoles on  $\Sigma \times S^1$ , where  $\Sigma$  is a closed Riemann surface, they showed that irreducible singular monopoles are bijective to  $\vec{t}$ -stable bundle pairs on  $\Sigma$ .

This is a monopole version of Kobayashi-Hitchin correspondence, which in general states that certain differential equations give rise to algebraic objects that satisfy suitable stability conditions [LT95] [Moc04].

To generalize such a correspondence on higher dimensional spaces, one needs to answer three questions:

- What is a higher dimensional monopole?
- What is its algebraic counterpart?
- What are the stability conditions?

In this paper, we are able to answer the first two questions, and we form a conjecture towards the answer of the third question.



We generalize monopole equation using dimension reduction from Hermite-Einstein equations. This generalization preserves most of the properties of three dimensional monopoles, in particular, when  $Y = S^1 \times X$ , where  $X$  is a smooth variety, any monopole on  $Y$  gives rise to a bundle pair on  $X$ , from which one can construct its spectral datum. With the insight from string theory [BHM07], we observe that the transition from monopole to its spectral datum is done directly by Fourier-Mukai transform.

This inspires us to apply Fourier-Mukai transform to monopole data on twisted  $S^1$  bundles over  $X$ , which gives rise to twisted spectral data, and we consider them to be the algebraic object in our generalization of Kobayashi-Hitchin correspondence. We prove two existence theorems for twisted spectral data: Theorem 4.4.3 states that twisted spectral data always exist on Riemann surface. Theorem 4.4.4 states that on a higher dimensional variety, with an assumption about the location of singularities, twisted spectral data exist.

Finally we form a conjecture about the stability conditions, and we expect that if the completion of spectral cover is irreducible, it extends to a stable twisted spectral datum. With the existence theorems, we can construct monopoles using algebraic geometry.

# Chapter 2

## Singular monopole

### 2.1 Monopole equation

Given a three dimensional Riemannian manifold  $Y$ , let  $E$  denote a rank  $n$  Hermitian vector bundle on  $Y$ , and let  $A$  be a unitary connection on  $E$ , let  $\phi$  be a skew Hermitian section of the bundle  $End(E)$ , the section  $\phi$  is called a Higgs field on  $E$  (with trivial coefficients). We say that the triple  $(E, A, \phi)$  satisfies Bogomolny equation if

$$F_A = *D_A\phi. \tag{2.1.1}$$

We first study the equation when  $Y = S^1 \times \Sigma$ , where  $\Sigma$  is a Riemann surface. In this case, Bogomolny equation comes from dimension reduction of the anti-self-duality equation: consider  $M = \mathbb{R} \times Y = \mathbb{R} \times S^1 \times \Sigma$ , fix a complex structure on  $\Sigma$ , let  $z$  be its local coordinate, we use  $s$  to parameterize  $\mathbb{R}$  and  $t$  to parameterize

$S^1$ , then  $w = s + it$  is a holomorphic coordinate on  $\mathbb{R} \times S^1$ , thus give  $M$  a complex structure so that  $M \cong \mathbb{C}^\times \times \Sigma$ . In terms of these coordinates, a connection form  $A$  on  $Y$  is expressed as

$$A = A_t dt + A_z dz + A_{\bar{z}} d\bar{z}$$

Given a Higgs field  $\phi$  on  $Y$ , let  $\hat{A} = \phi ds + A$  in this local trivialization, then  $\hat{A}$  glues into a connection on  $M$ , and we have

**Proposition 2.1.1.**  *$(A, \phi)$  satisfies Bogomolny equation if and only if  $\hat{A}$  satisfies anti-self-duality equation:*

$$*F_{\hat{A}} = -F_{\hat{A}}. \quad (2.1.2)$$

Fix a Kähler metric on  $\Sigma$ , and denote its Kähler form by  $\omega$ . We can extend the metric to a Kähler metric on  $M$ , and denote its Kähler form as  $\Omega$ . Split  $F_{\hat{A}}$  with respect to the complex structure on  $M$ , and denote by  $\Lambda F_{\hat{A}}$  the curvature component proportional to  $\Omega$ , we see that the anti-self-duality equation is equivalent to

$$F_{\hat{A}}^{0,2} = F_{\hat{A}}^{2,0} = 0, \quad \Lambda F_{\hat{A}} = 0. \quad (2.1.3)$$

A useful generalization of the anti-self-duality equation is Hermite-Einstein equation

$$F_{\hat{A}}^{0,2} = F_{\hat{A}}^{2,0} = 0, \quad \Lambda F_{\hat{A}} = \sqrt{-1}C \text{id}_E, \quad (2.1.4)$$

where  $C$  is a real constant.

To generalize Bogomolny's equation, we apply dimension reduction to the equation 2.1.4: let  $X$  be a  $k$  dimensional smooth projective variety, with Kähler form  $\omega$ ,

let  $Y$  be a  $S^1$  bundle over  $X$ , together with a Hermitian vector bundle  $E$  over  $Y$ , a connection  $A$  and a Higgs field  $\phi$ . In a local chart  $U$  of  $X$  where we can trivialize  $Y$  as  $S^1 \times U$ , use  $z_1, \dots, z_k$  for the holomorphic coordinates on  $U$ , and use  $t$  for the coordinate of  $S^1$ , suppose

$$\begin{aligned}\omega &= \sum_{1 \leq i, j \leq k} g_{i\bar{j}} dz_i \wedge d\bar{z}_j \\ A &= A_t dt + \sum_{j=1}^k (A_j dz_j + A_{\bar{j}} d\bar{z}_j)\end{aligned}$$

and denote

$$\nabla_j = \left( \frac{\partial}{\partial z_j} + A_j \right) dz_j, \quad 1 \leq j \leq k$$

$$\nabla_{\bar{j}} = \left( \frac{\partial}{\partial \bar{z}_j} + A_{\bar{j}} \right) d\bar{z}_j, \quad 1 \leq j \leq k$$

$$\nabla_t = \left( \frac{\partial}{\partial t} + A_t \right) dt$$

$$F_{ab} = [\nabla_a, \nabla_b], \quad a, b \in \{1, \dots, k, \bar{1}, \dots, \bar{k}, t\}$$

equation 2.1.4 is equivalent to:

$$\left\{ \begin{array}{l} F_{j\bar{l}} = 0, \quad F_{\bar{j}l} = 0 \\ F_{tj} = \sqrt{-1} \nabla_j \phi \wedge dt, \quad F_{t\bar{j}} = -\sqrt{-1} \nabla_{\bar{j}} \phi \wedge dt \\ g^{j\bar{l}} F_{j\bar{l}} - \nabla_t \phi = \sqrt{-1} C I \end{array} \right. \quad (2.1.5)$$

Notice that in the third equation of (2.1.5), we adopt the Einstein notation for the term  $g^{j\bar{l}} F_{j\bar{l}}$ , and do the sum only for the coefficients of the form, the same applies to  $\nabla_t \phi$ , which we actually mean  $\partial \phi / \partial t + [A_t, \phi]$ .

**Definition 2.1.2.** The triple  $(E, A, \phi)$  is a generalized monopole on  $Y$  if for each open set  $U \subset X$  where  $Y$  trivializes, equation 2.1.5 holds. The real constant  $C$  is called the slope of the monopole.

*Remark 2.1.3.* We will allow singular solutions, and in that case, we define generalized monopole on an open submanifold  $Y_0 \subset Y$  in the same way.

**Definition 2.1.4.** For a generalized monopole  $(E, A, \phi)$  on  $Y$  with slope  $C$ , we say that a generalized monopole  $(E', A', \phi')$  on  $Y$  with slope  $C'$  is a sub-monopole of  $(E, A, \phi)$  if  $E'$  is a sub-bundle of  $E$  preserved by  $A$  and  $\phi$ , and  $A|_{E'} = A'$ ,  $\phi|_{E'} = \phi'$ .

*Remark 2.1.5.* A sub-monopole must have the same slope, i.e.,  $C' = C$ .

**Definition 2.1.6.** A generalized monopole  $(E, A, \phi)$  on  $Y$  is irreducible if its only sub-monopoles are the trivial ones, i.e.  $(0, 0, 0)$  and  $(E, A, \phi)$ .

In local coordinates, denote

$$\nabla_U^{0,1} = \sum_{j=1}^k \nabla_{\bar{j}}$$

then the equation 2.1.5 is equivalent to

$$\begin{cases} [\nabla_U^{0,1}, \nabla_U^{0,1}] = 0 \\ [\nabla_U^{0,1}, \nabla_t - \sqrt{-1}\phi] = 0 \\ g^{j\bar{l}} F_{j\bar{l}} - \nabla_t \phi = \sqrt{-1}CI \end{cases} \quad (2.1.6)$$

We can think  $E$  as a family of vector bundles on  $U$ , parameterized by  $S^1$ . Then fix a  $t \in S^1$ , let  $\mathcal{E}_t = E|_{\{t\} \times U}$ , the first equation in 2.1.6 implies that  $\nabla_U^{0,1}$  defines an integrable complex structure on  $\mathcal{E}_t$ .

**Definition 2.1.7.** The scattering map from  $t_0$  to  $t_1$

$$R_{t_0, t_1} : \mathcal{E}_{t_0} \rightarrow \mathcal{E}_{t_1}$$

is defined by parallel transport with respect to  $A_t - i\phi$ , namely, for each  $\sigma_0 \in E|_{(t_0, z)}$  there is a unique solution  $\sigma(t) \in E|_{(t, z)}$  of

$$\frac{d\sigma}{dt} + (A_t - i\phi)\sigma = 0$$

with initial value  $\sigma(t_0) = \sigma_0$ . We set  $R_{t_0, t_1}(\sigma_0) = \sigma(t_1)$ .

According to the second equation in 2.1.6, for a smooth monopole, the scattering map defines a holomorphic isomorphism from  $\mathcal{E}_{t_0}$  to  $\mathcal{E}_{t_1}$ .

## 2.2 Dirac singularity

We consider solutions with Dirac type singularities, the prototype is a family of  $U(1)$ -monopoles on the three dimensional manifold

$$B_0^3(\epsilon) := \{(r, \theta, \psi) \mid 0 < r < \epsilon, \theta \in [0, \pi], \psi \in [0, 2\pi)\}$$

with the standard volume form on  $B_0^3(\epsilon)$ :

$$d\mu = r^2 \sin \theta \, dr \wedge d\theta \wedge d\psi.$$

Projecting along radial direction yields a map from  $B_0^3(\epsilon)$  to the unit sphere  $S^2$ , denote by  $L_k$  the Hermitian line bundle on  $B_0^3(\epsilon)$  obtained by pulling the line bundle of degree  $k$  on  $S^2$ . Explicitly, we can trivialize line bundles on the open sets

$$U_0 = \{\theta \neq \pi\}, \quad U_\pi = \{\theta \neq 0\}$$

and glue them into  $L_k$  via the transition function given by  $g_{\pi 0} = e^{\sqrt{-1}k\psi}$ . Consider the connection form  $A$  on  $L_k$  defined by

$$A_0 = \frac{\sqrt{-1}k}{2}(\cos \theta + 1) d\psi \text{ on } U_0,$$

$$A_\pi = \frac{\sqrt{-1}k}{2}(\cos \theta - 1) d\psi \text{ on } U_\pi,$$

on the overlap these forms satisfy

$$A_\pi + g_{\pi 0}^{-1} d g_{\pi 0} = A_0,$$

so  $A$  is a  $U(1)$ -connection on  $L_k$ . In this case, the Higgs field will be a function taking value in purely imaginary numbers, and if we choose

$$\phi = \frac{\sqrt{-1}k}{2r},$$

it is easy to verify that  $(L_k, A, \phi)$  satisfies Bogomolny equation on  $B_0^3(\epsilon)$ . Such a solution has boundary behavior that has a nice geometric interpretation, it is first explored in the paper [KW06], Witten and Kapustin showed that scattering map cross Dirac type singularities plays a role of Hecke modification in geometric Langlands program.

We extend this picture to higher dimensions. Let  $Y$  be an  $m$ -dimensional Riemannian manifold,  $Z \subset Y$  is a codimension 3 compact sub-manifold. For every  $p \in Z$ , we can choose local coordinates  $(x_1, \dots, x_m)$  so that:

- $p = (0, \dots, 0)$ ,
- $Z$  is given by  $x_1 = x_2 = x_3 = 0$ ,

- the metric is of the form  $I_m + O(R)$  as  $R := \sqrt{x_1^2 + \cdots + x_m^2} \rightarrow 0$ .

we call  $(x_1, \cdots, x_m)$  regular system of coordinates of  $Y$  with respect to  $Z$  centered at  $p$ .

**Definition 2.2.1.** With the above settings, let  $(E, A, \phi)$  be a generalized  $U(n)$ -monopole on  $Y$ , we say the monopole has a Dirac singularity along  $Z$  with weights  $(k_1, \cdots, k_n)$  if for every  $p \in Z$ , under every regular coordinates  $(x_1, \cdots, x_m)$  of  $Y$  with respect to  $Z$  centered at  $p$ ,

1. There is a unitary isomorphism  $\alpha$  of the restriction of the bundle  $E$  to

$$B_\epsilon := \{(x_1, x_2, x_3, 0, \cdots, 0) \mid 0 < R < \epsilon\} \cong B_0^3(\epsilon)$$

with a direct sum of line bundles  $L_{k_1} \oplus L_{k_2} \oplus \cdots \oplus L_{k_n}$  on  $B_0^3(\epsilon)$ .

2. Identify  $B_\epsilon$  with  $B_0^3(\epsilon)$ , and identify  $E$  with direct sum of vector bundle  $L_{k_1} \oplus L_{k_2} \oplus \cdots \oplus L_{k_n}$  on  $B_0^3(\epsilon)$  under the bundle isomorphism  $\alpha$ , in the trivializations of  $E$  over the two open subsets  $\{\theta \neq 0\}$  and  $\{\theta \neq \pi\}$  induced by the standard trivializations of the line bundles  $L_{k_i}$  the trivializations have transition function  $\text{diag}(e^{\sqrt{-1}k_1\psi}, \cdots, e^{\sqrt{-1}k_n\psi})$ , in both of the trivializations,  $\phi$  and  $A$  have asymptotic behaviors as follow:

$$\phi = \frac{\sqrt{-1}}{2r} \text{diag}(k_1, \cdots, k_n) + O(1), \quad D_A(r\phi) = O(1) \quad \text{as } r \rightarrow 0.$$



# Chapter 3

## Charbonneau-Hurtubise theorem

In the paper [CH11] Charbonneau and Hurtubise studied generalized monopoles of Dirac singularity on a product of a circle and a Riemann surface, we will summarize their results in this chapter. Let  $\Sigma$  denote a compact Riemann surface and fix a Kähler metric on it, with Kähler form  $\omega$ . Consider monopoles on  $Y = S^1 \times \Sigma$ . In this case, the monopole equations 2.1.5 can be written simply as:

$$F_A - \sqrt{-1}C\mathrm{id}_E\omega = *D_A\phi \tag{3.0.1}$$

and the singularities occur at a discrete set of points.

### 3.1 The scattering map

Now parameterize  $S^1 = \mathbb{R}/\mathbb{Z}$  by  $t \in [0, 1]$ , and consider monopole  $(E, A, \phi)$  with singularities at  $p_j$ ,  $j = 1, \dots, N$ . Assume  $p_j = (t_j, z_j) \in S^1 \times \Sigma$  has weights

$\vec{k}_j = (k_{j1}, k_{j2}, \dots, k_{jn})$ , furthermore, we require that  $0 < t_1 < \dots < t_N < 1$ . Write  $\mathcal{E}_t = E_{\{t\} \times \Sigma}$ . Regarding the scattering map  $R_{t,t'}$ , we have:

**Proposition 3.1.1.** *Away from  $t_i$ ,  $\mathcal{E}_t$  is a holomorphic bundle on  $\Sigma$ , and*

1. *If there is no singular parameter  $t_i$  between  $t$  and  $t'$ , the scattering map*

$$R_{t,t'} : \mathcal{E}_t \rightarrow \mathcal{E}_{t'}$$

*is an holomorphic isomorphism.*

2. *If only one  $t_i$  lies in between  $t$  and  $t'$ , then  $c_1(\mathcal{E}_{t'}) - c_1(\mathcal{E}_t) = \text{Tr}(\vec{k}_i)$ , where*

$$\text{Tr}(\vec{k}_i) = \sum_{j=1}^n k_{ij},$$

*the map  $R_{t,t'}$  is a meromorphic map which is an isomorphism away from  $z_i$ ,*

*and near  $z_i$  there exist trivializations of  $\mathcal{E}_t, \mathcal{E}_{t'}$  such that  $R_{t,t'}$  is given by*

$$\text{diag}((z - z_i)^{k_{i1}}, \dots, (z - z_i)^{k_{in}}).$$

Consider the map  $R_{0,1} : \mathcal{E}_0 \rightarrow \mathcal{E}_1$ , since  $\mathcal{E}_1 = \mathcal{E}_0$ , we get a meromorphic endomorphism of  $\mathcal{E}_0$  which is holomorphic outside of  $z_i$ 's.

**Definition 3.1.2.** A bundle pair  $(\mathcal{E}, \rho)$  consists of a holomorphic bundle  $\mathcal{E}$  on  $\Sigma$  and a meromorphic endomorphism  $\rho : \mathcal{E} \rightarrow \mathcal{E}$  such that  $\rho$  is an isomorphism outside of a finite set of points.

Thus we get a map from generalized monopoles to bundle pairs:

$$\Theta : (E, A, \phi) \mapsto (\mathcal{E}_0, R_{0,1})$$

In general, consider an arbitrary pair  $(\mathcal{E}, \rho)$  consisting of a rank  $n$  holomorphic vector bundle  $\mathcal{E}$  over  $\Sigma$ , and a meromorphic bundle automorphism  $\rho : \mathcal{E} \rightarrow \mathcal{E}$ . Near a singular point  $p$  of  $\rho$ , choose a coordinate  $z$  centered at  $p$  and a trivialization of  $\mathcal{E}$ , Iwahori et al. proved in [IM65]:

**Proposition 3.1.3.** *There are invertible holomorphic  $n \times n$  matrices  $F(z), G(z)$  and integers  $\vec{k} = \{k_1, \dots, k_n\}$ , such that in this trivialization,*

$$\rho = F(z) \operatorname{diag}(z^{k_1}, \dots, z^{k_n}) G(z)$$

*and the set of integers  $\{k_1, \dots, k_n\}$  is independent of the choice of  $F$  and  $G$ .*

**Definition 3.1.4.** In the above setting, we say that the bundle pair  $(\mathcal{E}, \rho)$  has singularity type  $\vec{k}$  at  $p$ .

According to Proposition 3.1.3, the bundle pair  $(\mathcal{E}_0, R_{0,1})$  obtained from above has singularity type  $\vec{k}_i$  at  $z_i$ .

Notice that by proposition 3.1.1, we have

$$c_1(\mathcal{E}_1) - c_1(\mathcal{E}_0) = \sum_{j=1}^N \operatorname{Tr}(\vec{k}_j),$$

since  $\mathcal{E}_1 = \mathcal{E}_0$ , a bundle pair corresponding to a generalized monopole must satisfy

$$\sum_{i=1}^N \operatorname{Tr}(\vec{k}_i) = 0.$$

**Proposition 3.1.5.** *For a singular  $U(n)$ -monopole with slope  $C$ , we have*

$$\sum_{i=1}^N \operatorname{Tr}(\vec{k}_i) t_i = c_1(\mathcal{E}_0) + \frac{nC}{2\pi} \operatorname{vol}(\Sigma).$$

## 3.2 Stability

Consider a bundle pair  $(\mathcal{E}, \rho)$  on  $\Sigma$ , with singular type  $\vec{k}_i$  at  $z_i$ ,  $i = 1, 2, \dots, N$ . Given a list of numbers  $\vec{t} = (t_1, t_2, \dots, t_N) \in T^N$ , where  $T^N$  denotes the torus formed by the product of  $N$  circles of circumference 1, for the bundle pair  $(\mathcal{E}, \rho)$ , we define

**Definition 3.2.1.**  $\vec{t}$ -degree

$$\delta_{\vec{t}}(\mathcal{E}, \rho) = c_1(\mathcal{E}) - \sum_{j=1}^N t_j \operatorname{Tr}(\vec{k}_j),$$

and  $\vec{t}$ -slope

$$\mu_{\vec{t}}(\mathcal{E}, \rho) = \delta_{\vec{t}}(\mathcal{E}, \rho) / \operatorname{rank} \mathcal{E}.$$

**Definition 3.2.2.** A bundle pair  $(\mathcal{E}, \rho)$  is  $\vec{t}$ -stable if any proper non-trivial  $\rho$ -invariant subbundle has a strictly smaller  $\vec{t}$ -slope.

Let  $(E, A, \phi)$  be an irreducible  $U(n)$ -monopole on  $Y$ , with singularity  $p_i = (t_i, z_i) \in S^1 \times \Sigma$ ,  $i = 1, \dots, N$ , let  $\vec{t} = (t_1, \dots, t_N)$ , Charbonneau and Hurtubise proved the following two theorems:

**Theorem 3.2.3.** *If  $(\mathcal{E}, \rho)$  is the image of  $(E, A, \phi)$  under the map  $\Theta$ , then it is  $\vec{t}$ -stable.*

**Theorem 3.2.4.** *Given a  $\vec{t}$ -stable bundle pair  $(\mathcal{E}, \rho)$  on  $\Sigma$ , with singularity type  $\vec{k}_i$  at  $z_i$ ,  $i = 1, 2, \dots, N$ , such that*

$$\sum_{i=1}^N \operatorname{Tr}(\vec{k}_i) = 0.$$

Let  $p_i = (t_i, z_i) \in S^1 \times \Sigma$ , there is a generalized monopole  $(E, A, \phi)$  on  $S^1 \times \Sigma$  with Dirac-type singularities of weight  $\vec{k}_i$  at  $p_i$ , for which  $\Theta(E, A, \phi) = (\mathcal{E}, \phi)$ .

*Remark 3.2.5.* Given  $\vec{k}_i$ ,  $1 \leq i \leq N$ , such that

$$\sum_{i=1}^N \text{Tr}(\vec{k}_i) = 0,$$

denote by

$$\mathcal{M}_{k_0}^{ir}(S^1 \times \Sigma, p_1, \dots, p_N, \vec{k}_1, \dots, \vec{k}_N)$$

the moduli of irreducible  $U(n)$ -monopoles on  $S^1 \times \Sigma$  with Dirac singularity of type  $\vec{k}_i$  at  $p_i = (t_i, z_i)$ , such that  $\mathcal{E}_0$  has degree  $k_0$ , and denote by

$$\mathcal{M}_s(\Sigma, k_0, z_1, \dots, z_N, \vec{k}_1, \dots, \vec{k}_N, \vec{t})$$

the moduli of  $\vec{t}$ -stable bundle pairs with singularity of type  $\vec{k}_i$  at  $z_i$ , such that  $\mathcal{E}$  is a holomorphic rank  $n$  vector bundle of degree  $k_0$  on  $\Sigma$ . Charbonneau-Hurtubise theorem implies the map

$$\Theta : \mathcal{M}_{k_0}^{ir}(S^1 \times \Sigma, p_1, \dots, p_N, \vec{k}_1, \dots, \vec{k}_N) \rightarrow \mathcal{M}_s(\Sigma, k_0, z_1, \dots, z_N, \vec{k}_1, \dots, \vec{k}_N, \vec{t})$$

$$(E, A, \phi) \mapsto (\mathcal{E}_0, R_{0,1})$$

is a bijection.

# Chapter 4

## Spectral data

When  $Y$  is an  $S^1$  bundle over  $X$ , we can consider the scattering map locally. In this case the bundle pairs do not glue into a global one, in fact, they glue only into a twisted pair. To understand such pairs it is often convenient to consider their spectral data [DM96].

### 4.1 Spectral datum of a bundle pair

Consider a bundle pair  $(\mathcal{E}, \rho)$  over a quasi-projective variety  $U$ , let  $Z_1, \dots, Z_N$  be its singular loci. We can associate to it a pair  $(\tilde{U}, \mathcal{L})$ , where  $\tilde{U}$  is a branched cover of  $U_0 := U \setminus \{Z_1, \dots, Z_N\}$  and  $\mathcal{L}$  is a sheaf on  $\tilde{U}$ .

**Definition 4.1.1.** The spectral cover  $\tilde{U}$  is defined to be

$$\tilde{U} = \{(\lambda, z) \in \mathbb{C}^\times \times U_0 \mid \det(\lambda I - \rho(z)) = 0\},$$

where  $I \in \text{End}(\mathcal{E})$  is identity over each  $z \in U$ .

**Definition 4.1.2.** Let  $p : \mathbb{C}^\times \times U \rightarrow U$  be the natural projection, the spectral sheaf  $\mathcal{L}$  is a subsheaf of  $p^*\mathcal{E}$  restricted on  $\tilde{U}$  which assigns to  $(\lambda, z)$  the eigenspace of  $\ker(p^*\rho(z) - \lambda \cdot \text{id})$ .

If  $\rho(z)$  is regular everywhere (i.e. it has distinct eigenvalues for different Jordan blocks), then  $\mathcal{L}$  is a line bundle on  $\tilde{U}$ . Note that being nonregular is a complex codimension three condition, and hence when  $\dim_{\mathbb{C}} U \leq 2$ , a generic  $\rho$  is regular.

The spectral datum  $(\tilde{U}, \mathcal{L})$  reconstructs the bundle pair  $(\mathcal{E}, \rho)$  away from its singularities. Indeed, if we denote by  $\pi : \tilde{U} \rightarrow U_0$  the restriction of  $p$  on  $\tilde{U}$ , and if  $\eta$  is the tautological section of  $p^*(\mathbb{C}^\times \times U) \cong \mathbb{C}^\times \times \mathbb{C}^\times \times U$  consisting of points  $(\lambda, \lambda, z)$  in local coordinates, then

$$\pi_*\mathcal{L} = \mathcal{E}|_{U_0}, \quad \pi_*\eta = \rho.$$

Observe that over  $U_0$ , different choices of trivialization yield conjugate bundle maps  $\rho$ , and therefore they have the same spectral cover, and all these  $\tilde{U}$  glue into a global cover of  $X$ .

**Definition 4.1.3.** Let  $X$  be a smooth projective variety, we say that  $\tilde{X}$  is a spectral cover of  $X$  if there exist a matrix of meromorphic function  $\rho$  on  $X$ , such that

$$\tilde{X} = \{(z, \lambda) \in X \times \mathbb{C}^\times \mid \det(\lambda I - \rho(z)) = 0\}.$$

Notice that although spectral covers glue, spectral sheaves do not, in fact, they

glue into a twisted object, and this can be better understood using a Fourier-Mukai transform.

## 4.2 Fourier-Mukai transform

For monopoles without singularities, the conversion from monopole data to their spectral data can be constructed directly, via Fourier-Mukai transform. In general, Fourier-Mukai transform for real tori, as defined in [AP01] and in [BMP01], identifies local systems on a real torus  $T$  with skyscraper sheaves of finite length on the dual torus  $\widehat{T}$ . In our case, we need to extend  $\widehat{T}$  to a complex affine torus.

Let  $\Lambda_1 = \mathbb{Z}^m$ ,  $\Lambda_2 = \mathbb{Z}^n$  be lattices, consider

$$\Lambda = \Lambda_1 \oplus \Lambda_2, \quad V = \Lambda_1 \otimes_{\mathbb{Z}} \mathbb{R} \oplus \Lambda_2 \otimes_{\mathbb{Z}} \mathbb{C},$$

then  $T = V/\Lambda \cong (S^1)^m \times (\mathbb{C}^\times)^n$  is a mixed (real and complex) torus. Let  $\text{Pic}(T)$  be the isomorphism class of  $\mathbb{C}^\times$ -bundles on  $T$ . We define a group  $P(\Lambda)$  whose elements are pairs  $(A, \chi)$ , where  $A \in \text{Alt}^2(\Lambda, \mathbb{Z})$  is an alternating 2-form on  $\Lambda$ , and  $\chi$  is a semicharacter for  $A$ , i.e., a map  $\chi : \Lambda \rightarrow \mathbb{C}^\times$ , such that

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{\sqrt{-1}\pi A(\lambda, \mu)}, \quad \forall \lambda, \mu \in \Lambda.$$

The group structure of  $P(\Lambda)$  is given by

$$(A_1, \chi_1)(A_2, \chi_2) = (A_1 + A_2, \chi_1\chi_2).$$

The form  $A$  is identified with the first Chern class via  $\text{Alt}^2(\Lambda, \mathbb{Z}) \cong H^2(T, \mathbb{Z})$ ,



and we have the exact sequence:

$$0 \rightarrow \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^\times) \rightarrow \text{Pic}(T) \xrightarrow{c_1} H^2(T, \mathbb{Z}) \rightarrow 0.$$

The kernel  $\text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^\times)$  is identified with flat  $\mathbb{C}^\times$ -bundles on  $T$ .

Given a line bundle  $L$  on  $T$  corresponding to  $(A, \chi) \in P(\Lambda)$ , the factor of automorphy of  $L$  is defined as a function  $\alpha_L : V \times \Lambda \rightarrow \mathbb{C}^\times$ , such that

$$\alpha_L(x, \lambda) = \chi(\lambda) e^{\sqrt{-1}\pi A(x, \lambda)}.$$

Here we extend  $A$  to  $V \times V$  to be  $\mathbb{C}$ -linear. Then the global sections of  $L$  are described by the smooth functions  $s : V \rightarrow \mathbb{C}$ , satisfying

$$s(x + \lambda) = \alpha_L(x, \lambda) s(x), \quad \forall x \in V, \lambda \in \Lambda.$$

Now we consider flat line bundles on  $S^1$ , for every  $z \in \mathbb{C}$ , there is a flat line bundle  $L_z$  on  $S^1$  whose associated pair is

$$A_z = 0, \quad \chi_z(\lambda) = e^{2\sqrt{-1}\pi \lambda z}.$$

On  $S^1 \times \mathbb{C}^\times$ , we define the Poincaré bundle  $\mathcal{P}$  to be the line bundle associated with the pair  $(A, \chi) \in P(\mathbb{Z} \times \mathbb{Z})$ , such that

$$A((\lambda_1, \mu_1), (\lambda_2, \mu_2)) = \mu_1 \lambda_2 - \mu_2 \lambda_1, \quad \chi(\lambda, \mu) = e^{\sqrt{-1}\pi \lambda \mu}.$$

The corresponding factor of automorphy is

$$\alpha_{\mathcal{P}}(t, z, \lambda, \mu) = e^{\sqrt{-1}\pi(\lambda z - \mu t + \mu \lambda)}.$$

We can apply the automorphism induced by

$$\phi : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{C}^\times, \quad \phi(t, z) = e^{\sqrt{-1}\pi tz},$$

then the factor of automorphy is changed to

$$\alpha'_{\mathcal{P}}(t, z, \lambda, \mu) = e^{2\sqrt{-1}\pi\lambda z}. \quad (4.2.1)$$

Thus  $\mathcal{P}_{|S^1 \times \{z\}} \cong L_z$ . We define the connection  $\nabla_{\mathcal{P}}$  on  $\mathcal{P}$  to have connection form  $A_{\mathcal{P}} = 2\sqrt{-1}\pi z dt$  in the gauge where the factor of automorphy of  $\mathcal{P}$  is the one in (4.2.1).

Denote by  $p_1, p_2$  the projections onto the two factors of  $S^1 \times \mathbb{C}^\times$ . Let

$$\Omega^{m,n} = p_1^* \Omega_{S^1}^m \otimes_{C_{S^1 \times \mathbb{C}^\times}^\infty} p_2^* \Omega_{\mathbb{C}^\times}^n.$$

the connection  $\nabla_{\mathcal{P}}$  splits into two operators:

$$\lambda_1 : \mathcal{P} \rightarrow \mathcal{P} \otimes \Omega^{1,0}, \quad \nabla_2 : \mathcal{P} \rightarrow \mathcal{P} \otimes \Omega^{0,1}.$$

In the gauge of equation (4.2.1), the action on  $\nabla_1, \nabla_2$  on sections has the form

$$\nabla_1 s = \frac{\partial s}{\partial t} dt + 2\sqrt{-1}\pi z s dt.$$

Given a local system  $(E, \nabla)$  on  $S^1$ , let  $\mathcal{E}$  be the sheaf of sections of  $E$ . We pull back  $(\mathcal{E}, \nabla)$  to  $S^1 \times \mathbb{C}^\times$  and couple with the pair  $(\mathcal{P}, \nabla_1)$  to get  $(p_1^* \mathcal{E} \otimes \mathcal{P}, \nabla_1^\mathcal{E})$ . It is shown in [BMP01] that  $R^0 p_{2*}(\ker \nabla_1^\mathcal{E}) = 0$  and  $R^1 p_{2*}(\ker \nabla_1^\mathcal{E})$  is a skyscraper sheaf of finith length. We define  $\mathcal{F}(E, \nabla) = R^1 p_{2*}(\ker \nabla_1^\mathcal{E})$ .

Notice that any flat vector bundle on  $S^1$  is a direct sum of flat line bundles, and for a flat line bundle  $L_z$ , its Fourier-Mukai image  $\mathcal{F}(L_z)$  is the skyscraper sheaf of

length one supported at  $e^{2\sqrt{-1}\pi z} \in \mathbb{C}^\times$ , which coincides with the holonomy of  $L_z$ . Therefore  $\mathcal{F}(E, \nabla)$  is its spectral datum.

Let  $U$  be a quasi-projective variety, we have a Poincaré line bundle  $\mathcal{P}$  with connection  $\nabla_{\mathcal{P}}$  on  $S^1 \times U \times \mathbb{C}^\times$ , which restricts on each  $S^1 \times \{u\} \times \mathbb{C}^\times$  to  $(\mathcal{P}, \nabla_{\mathcal{P}})$ . We apply the Fourier-Mukai transform fiberwise, this discussion together with the Charbonneau-Hurtubise theorem now give the following

**Proposition 4.2.1.** *The Fourier-Mukai transform with kernel  $(\mathcal{P}, \nabla_{\mathcal{P}})$  converts smooth monopole datum  $(E, A - \sqrt{-1}\phi)$  on  $S^1 \times U$  to its spectral datum  $(\tilde{U}, \mathcal{L})$ .*

On non-trivial  $S^1$  bundles, we apply the Fourier-Mukai transform locally and get twisted spectral data by gluing, this enables us to extend the theorem of Charbonneau and Hurtubise. We will interpret twisted spectral data as a geometric object – a line bundle on an analytic gerbe.

### 4.3 Analytic gerbe

Let  $X$  be a complex manifold and let  $\mathcal{O}_X$  denote the sheaf of holomorphic functions on  $X$ . We can consider a cohomology class in  $H^2(X, \mathcal{O}_X^\times)$  as equivalence class of  $\mathcal{O}_X^\times$ -gerbe, much in the same way as we interpret elements in  $H^1(X, \mathcal{O}_X^\times)$  as equivalence class of holomorphic line bundles. We first explain this in Čech cohomology:

**Definition 4.3.1** ([Hit01]). Given an open cover  $\{U_\alpha\}$  of  $X$ , an  $\mathcal{O}_X^\times$ -gerbe  $g$  is an assignment to each threefold intersection  $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$  an invertible

holomorphic function  $g_{\alpha\beta\gamma}$ , such that

$$g_{\alpha\beta\gamma} = g_{\alpha\gamma\beta}^{-1} = g_{\beta\alpha\gamma}^{-1} = g_{\gamma\beta\alpha}^{-1}$$

and on each fourfold intersection

$$\delta(g)_{\alpha\beta\gamma\delta} = g_{\beta\gamma\delta} \cdot g_{\alpha\gamma\delta}^{-1} \cdot g_{\alpha\beta\delta} \cdot g_{\alpha\beta\gamma}^{-1} = 1$$

**Definition 4.3.2.** A trivialization of gerbe  $g$  on  $U$  is defined by holomorphic functions

$$f_{\alpha\beta} = f_{\beta\alpha}^{-1} : U_{\alpha\beta} \cap U \rightarrow \mathbb{C}^*$$

on twofold intersections in  $U$  such that in  $U_{\alpha\beta\gamma} \cap U$ ,

$$g_{\alpha\beta\gamma} = f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha}$$

**Definition 4.3.3.** A gerbe is trivial if it has a global trivialization. Two gerbes are equivalent if their difference is trivial.

In this way, we consider  $H^2(X, \mathcal{O}_X^\times)$  as equivalence class of  $\mathcal{O}_X^\times$ -gerbes.

**Definition 4.3.4** ([GG71]). Let  $g$  be an  $\mathcal{O}_X^\times$ -gerbe, define a  $g$ -twisted coherent sheaf on  $X$  as a collection  $\{F_i, \phi_{ij}\}$  of coherent sheaves  $F_i$  of  $\mathcal{O}_X$ -modules on  $U_i$ , together with isomorphisms:

$$\phi_{ij} : F_{j|U_{ij}} \rightarrow F_{i|U_{ij}},$$

such that  $\phi_{ii} = \text{id}_{F_i}$ ,  $\phi_{ij} = \phi_{ji}^{-1}$  and  $\phi_{ij}\phi_{jk}\phi_{jk} = g_{ijk} \text{id}_{F_i}$ .

**Definition 4.3.5.** Given two  $g$ -twisted sheaves  $\mathcal{F} = \{F_i, \phi_{ij}\}$  and  $\mathcal{H} = \{H_i, \gamma_{ij}\}$ , a homomorphism  $\mathbf{f} : \mathcal{F} \rightarrow \mathcal{H}$  is a collection of  $\mathbf{f} = \{f_i\}$  of sheaf morphisms  $f_i : F_i \rightarrow H_i$ , such that  $f_i \circ \phi_{ij} = \gamma_{ij} \circ f_j$ .

Note that  $g$ -twisted coherent sheaves might not exist. For example, we can consider  $g$ -twisted vector bundles of rank  $n$ , then

$$\det(\phi_{ij}) \det(\phi_{jk}) \det(\phi_{ki}) = g_{ijk}^n,$$

therefore we get a trivialization of  $g^n$ , so  $g$ -twisted vector bundles exist only if  $g$  is  $n$ -torsion, and in particular,  $g$ -twisted line bundles only exist on trivial gerbes.

Geometrically, we can consider such gerbes as analytic stacks [DP03]: denote by  $B\mathcal{O}_X^\times$  the classifying stack of  $\mathcal{O}_X^\times$ , i.e., it assigns to each open set  $V$  a category, whose objects are  $\mathcal{O}_X^\times$ -torsors on  $V$  and morphisms are isomorphisms of torsors. Then an  $\mathcal{O}_X^\times$ -gerbe on  $X$  is a  $B\mathcal{O}_X^\times$  torsor on  $X$ , i.e. a stack of groupoids over  $X$ , which admits a principal homogeneous action of  $B\mathcal{O}_X^\times$ . In this way, a gerbe  $\mathcal{H}$  on  $X$  comes with a projection  $\mathcal{H} \rightarrow X$ , and is classified by an element in  $H^1(X, B\mathcal{O}_X^\times)$ . This is consistent with the previous discussion since  $B\mathcal{O}_X^\times = \mathcal{O}_X^\times[1]$  and so  $H^1(X, B\mathcal{O}_X^\times) = H^1(X, \mathcal{O}_X^\times[1])$  is isomorphic to  $H^2(X, \mathcal{O}_X^\times)$ . Thus for a gerbe  $g$ , we can assign to it a stack  ${}_gX$ . A  $g$ -twisted sheaf on  $X$  can be interpreted as a weight one sheaf on  ${}_gX$ , where a sheaf on  ${}_gX$  is a representation of the sheaf of groupoids  ${}_gX \rightarrow X$ .

## 4.4 Twisted spectral data

Let  $L \rightarrow X$  be a holomorphic line bundle with Hermitian metric and  $Y$  be its unit circle bundle, let  $(E, A, \phi)$  be a singular monopole on  $Y$ , with singularities located in the fiber over divisors  $Z_1, \dots, Z_N \subset X$ . Take an open cover  $\{U_i\}$  of  $X_0 := X \setminus \{Z_1, \dots, Z_N\}$ , so that for each  $U_i$  the fiber bundle  $Y_i := Y|_{U_i} \rightarrow U_i$  trivializes. Using this trivialization we construct Poincaré bundle with connection  $(\mathcal{P}_i, A_i)$  on  $Y_i \times \mathbb{C}^\times$  as in Proposition 4.2.1. The Fourier-Mukai transform with kernel  $(\mathcal{P}_i, A_i)$  converts the monopole  $(E, A, \phi)|_{Y_i}$  into spectral data on  $U_i \times \mathbb{C}^\times$ . The spaces  $U_i \times \mathbb{C}^\times$  glue into  $X \times \mathbb{C}^\times$ , the spectral cover glue into  $\tilde{X} \subset X_0 \times \mathbb{C}^\times$ , but the Poincaré bundle with connection  $(\mathcal{P}_i, A_i)$  does not glue, the obstruction is measured by the cohomology class:

$$[H] = \frac{1}{4\pi i} p_X^*(c_1(Y)) \wedge p_{\mathbb{C}^\times}^*(dz/z - d\bar{z}/\bar{z}) \in H^3(X \times \mathbb{C}^\times, \mathbb{Z}).$$

Since  $c_1(Y) = c_1(L)$  is a  $(1, 1)$ -form, we know that  $[H]$  is of type  $(1, 2) + (2, 1)$ , according to the exponential exact sequence:

$$\dots \rightarrow H^2(X \times \mathbb{C}^\times, \mathcal{O}^\times) \xrightarrow{\delta} H^3(X \times \mathbb{C}^\times, \mathbb{Z}) \xrightarrow{\alpha} H^3(X \times \mathbb{C}^\times, \mathcal{O}) \rightarrow \dots$$

since  $\alpha([H]) = 0$ , there is a gerbe represented by  $g \in H^2(X \times \mathbb{C}, \mathcal{O}^\times)$ , so that  $\delta(g) = [H]$ . We denote the corresponding stack by  $\mathcal{H}$ .

Tautologically the locally defined Poincaré line bundles with connections glue into a global pair  $(\mathcal{P}, A_{\mathcal{P}})$  on the gerbe  $Y \times_X \mathcal{H}$ , and we get the following:

**Proposition 4.4.1.** *The Fourier-Mukai transform associated to  $(\mathcal{P}, A_{\mathcal{P}})$  converts*

a monopole datum  $(E, A - \sqrt{-1}\phi)$  on  $Y$  to spectral datum  $(\tilde{X}, \mathcal{L})$ , where  $\mathcal{L}$  is a weight one coherent sheaf on the restriction of  $\mathcal{H}$  to  $\tilde{X}$ , or equivalently a trivialization of the gerbe  $\mathcal{H}|_{\tilde{X}}$ .

**Definition 4.4.2.** A twisted spectral datum on  $X$  with singularities along  $Z_1, \dots, Z_N$  is a triple  $(\mathcal{H}, \tilde{X}, \mathcal{L})$ , where  $\mathcal{H}$  is a non-trivial  $\mathcal{O}_X^\times$ -gerbe on  $X$ ,  $\tilde{X}$  is a spectral cover of  $X \setminus \{Z_1, \dots, Z_N\}$  and  $\mathcal{L}$  is a weight one coherent sheaf on  $\mathcal{H}|_{\tilde{X}}$ .

**Theorem 4.4.3.** *On a Riemann surface, twisted spectral data always exist.*

*Proof.* Let  $\Sigma$  be a Riemann surface,  $z_1, \dots, z_N \in \Sigma$ , and  $\mathcal{H}$  be a non-trivial gerbe on  $\Sigma \times \mathbb{C}^\times$ . Choose any spectral cover  $\tilde{\Sigma}$  of  $\Sigma \setminus \{z_1, \dots, z_N\}$ , by exponential exact sequence:

$$\dots \rightarrow H^2(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}) \rightarrow H^2(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}^\times) \rightarrow H^3(\tilde{\Sigma}, \mathbb{Z}) \rightarrow \dots$$

we have  $H^2(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}^\times) = 0$  since the terms before and after it are 0. Therefore the restriction of gerbe  $\mathcal{H}$  on  $\tilde{\Sigma}$  is always trivial. Any trivialization of  $\mathcal{H}$  gives rise to a weight one line bundle, therefore twisted spectral data exist.  $\square$

Let  $X$  be a smooth complex projective variety of complex dimension  $k$ , and let  $Z_1, \dots, Z_N$  be divisors on  $X$ , we have:

**Theorem 4.4.4.** *If there are non-negative integers  $b_1, \dots, b_N$ , such that  $D = b_1 Z_1 + \dots + b_N Z_N$  is ample, then twisted spectral data with singularities along  $Z_1, \dots, Z_N$  exist.*

*Proof.* Denote by

$$X_0 = X \setminus \{Z_1, \dots, Z_N\}, \quad M = X \times \mathbb{C}^\times, \quad M_0 = X_0 \times \mathbb{C}^\times.$$

Let  $\omega \in H_{dR}^2(X, \mathbb{Z})$  represent the Poincaré dual of  $D$ , and let  $\tau$  represent the generator of  $H_{dR}^1(\mathbb{C}^\times, \mathbb{Z})$ , denote by  $p_1, p_2$  the natural projections from  $M$  to  $X$  and to  $\mathbb{C}^\times$  respectively, let

$$p_1^* \omega \wedge p_2^* \tau \in H^3(M, \mathbb{Z})$$

be the image of gerbe  $\mathcal{H} \in H^2(M, \mathcal{O}_M^\times)$ .

Consider the map

$$\alpha : H^2(M, \mathcal{O}_M^\times) \rightarrow H^2(M_0, \mathcal{O}_{M_0}^\times),$$

we claim that  $\alpha(\mathcal{H}) = 0$ .

Notice that we can fit  $\alpha$  into exponential exact sequence:

$$\begin{array}{ccc} H^2(M, \mathcal{O}_M) & \longrightarrow & H^2(M_0, \mathcal{O}_{M_0}) \\ \downarrow & & \downarrow \\ H^2(M, \mathcal{O}_M^\times) & \xrightarrow{\alpha} & H^2(M_0, \mathcal{O}_{M_0}^\times) \\ \downarrow \delta_1 & & \downarrow \delta_2 \\ H^3(M, \mathbb{Z}) & \xrightarrow{\beta} & H^3(M_0, \mathbb{Z}) \end{array}$$

and  $\delta_1(\mathcal{H}) = p_1^* \omega \wedge p_2^* \tau$ .

We know that

$$H^i(\mathbb{C}^\times, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, 1 \\ 0, & i > 1 \end{cases}$$



By Kunneth formula for  $M = X \times \mathbb{C}^\times$ ,

$$H^3(M, \mathbb{Z}) \cong H^3(X, \mathbb{Z}) \otimes H^0(\mathbb{C}^\times, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \otimes H^1(\mathbb{C}^\times, \mathbb{Z}),$$

therefore

$$\delta_1(\mathcal{H}) \cong \omega \otimes \tau \in H^2(X, \mathbb{Z}) \otimes H^1(\mathbb{C}^\times, \mathbb{Z})$$

and it maps to  $\omega$  under the natural isomorphism

$$H^2(X, \mathbb{Z}) \otimes H^1(\mathbb{C}^\times, \mathbb{Z}) \cong H^2(X, \mathbb{Z}).$$

Similarly, we have

$$H^3(M_0, \mathbb{Z}) \cong H^3(X_0, \mathbb{Z}) \otimes H^0(\mathbb{C}^\times, \mathbb{Z}) \oplus H^2(X_0, \mathbb{Z}) \otimes H^1(\mathbb{C}^\times, \mathbb{Z}),$$

and

$$\beta \circ \delta_1(\mathcal{H}) \in H^2(X_0, \mathbb{Z}) \otimes H^1(\mathbb{C}^\times, \mathbb{Z}) \cong H^2(X_0, \mathbb{Z}).$$

Furthermore, one can check that  $\beta \circ \delta_1(\mathcal{H}) = \sigma(\omega)$ , where  $\sigma$  is the restriction map

$$\sigma : H^2(X, \mathbb{Z}) \rightarrow H^2(X_0, \mathbb{Z}).$$

Let  $W = Z_1 \cup \dots \cup Z_N$ , since  $X_0 = X \setminus W$ , the map  $\sigma$  fits into the Gysin sequence:

$$H^0(W, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z}) \rightarrow H^2(X_0, \mathbb{Z}) \rightarrow \dots \quad (4.4.1)$$

its Poincaré dual gives:

$$H_{2k-2}(W, \mathbb{Z}) \xrightarrow{\iota} H_{2k-2}(X, \mathbb{Z}) \xrightarrow{\gamma} H_{2k-2}(X_0, \mathbb{Z}) \rightarrow \dots$$

Since the Pioncaré dual of  $\omega$  is represented by the divisor  $D$ , and  $D$  is spanned by irreducible components of  $W$ , so  $D$  is in the image of  $\iota : H_{2k-2}(W, \mathbb{Z}) \rightarrow H_{2k-2}(X, \mathbb{Z})$ , hence  $\gamma : H_{2k-2}(X, \mathbb{Z}) \rightarrow H_{2k-2}(X_0, \mathbb{Z})$  sends  $D$  to 0. Go back to the sequence 4.4.1, we see that  $\sigma(\omega) = 0$ , therefore  $\beta \circ \delta_1(\mathcal{H}) = 0$ .

Now consider  $\alpha(\mathcal{H})$ , since

$$\delta_2 \circ \alpha(\mathcal{H}) = \beta \circ \delta_1(\mathcal{H}) = 0,$$

$\alpha(\mathcal{H})$  comes from  $H^2(M_0, \mathcal{O}_{M_0})$ . Because  $D$  is ample, and  $D$  is a non-negative integer combination of  $Z_i$ 's, we know that  $X_0$  is a Stein manifold, thus  $H^i(X_0, \mathcal{O}_{X_0}) = 0$  for  $i > 0$ , then by Kunneth formula,  $H^2(M_0, \mathcal{O}_{M_0}) = 0$ . Hence the pre-image of  $\alpha(\mathcal{H})$  in  $H^2(M_0, \mathcal{O}_{M_0})$  can only be 0, so  $\alpha(\mathcal{H})$  is 0 as well.

Therefore the gerbe  $\mathcal{H}$  restricts to a trivial gerbe on  $\tilde{X}$ , twisted spectral data exist. □

In the above two theorems, the existence of twisted spectral data relies on the property that the restriction of a gerbe to the spectral cover is trivial. If the restriction fails to be trivial, although there is no twisted line bundle, there is still a chance of getting twisted sheaves at higher rank.

Let  $L$  and  $\mathcal{H}$  be given as above. These data give a quotient presentation[DP03] of  $\mathcal{H}$ . Define the bundle of groups

$$G = \coprod_{n \in \mathbb{Z}} (L^{\otimes n})^\times,$$

it fits into the exact sequence:

$$0 \rightarrow \underline{\mathbb{C}}^\times \rightarrow G \rightarrow \underline{\mathbb{Z}} \rightarrow 0$$

where  $\underline{\mathbb{C}}^\times$  and  $\underline{\mathbb{Z}}$  are trivial bundles over  $X$ . Let  $G$  act on  $\underline{\mathbb{C}}$  via its projection to  $\underline{\mathbb{Z}}$ . The stablizer of this action at each point is isomorphic to  $\mathbb{C}^\times$ . We can consider  $X \times \mathbb{C}^\times = \underline{\mathbb{C}}/\underline{\mathbb{Z}}$ , then the gerbe  $\mathcal{H}$  is the quotient stack  $[\underline{\mathbb{C}}/G]$ . A level one sheaf on  $\mathcal{H}_{|\tilde{X}}$  is a  $G \times_X \tilde{X}$ -equivariant sheaf on  $\tilde{X} \times \mathbb{C}$  on which  $\mathbb{C}^\times$  acts by its tautological character. This gives a concrete way of constructing many spectral line bundles.

# Chapter 5

## Conjecture

In order to obtain a monopole version of Kobayashi-Hitchin correspondence, we need suitable stability conditions on the algebraic data. As Charbonneau and Hurtubise have shown, in the non-twisted case one can use  $\vec{t}$ -stability for a bundle pair. We can interpret  $\vec{t}$ -degree as a parabolic degree.

**Definition 5.0.5** ([IS08]). Given a vector bundle  $E$  on  $X$ , let  $D$  be an effective divisor on  $X$ , a parabolic structure on  $E$  corresponding to  $D$  is a collection of vector bundles  $E_i$ , with a filtration as sheaves of  $\mathcal{O}_X$ -modules:

$$E = E_1 \supset E_2 \supset \cdots \supset E_\ell \supset E_{\ell+1} = E(-D)$$

together with a system of parabolic weights:

$$0 \leq \alpha_1 < \cdots < \alpha_\ell < 1.$$

Its parabolic degree is

$$\text{pardeg } E = \sum_{j=1}^{\ell+1} (\alpha_j - \alpha_{j-1}) \deg E_j,$$

here  $\alpha_0 = 0, \alpha_{\ell+1} = 1$ .

Consider a bundle pair  $(\mathcal{E}, \rho)$  on  $\Sigma$ , let  $\rho$  be singular at  $z_i$  of type  $\vec{k}_i = (k_{i1}, \dots, k_{in})$ , let  $k_{ij}^+ = \max(k_{ij}, 0)$  and  $k_{ij}^- = \min(k_{ij}, 0)$ . In a punctured neighborhood of  $z_i$ , we can trivialize  $\mathcal{E}$  such that

$$\rho = F(z) \text{diag}((z - z_i)^{k_{i1}}, \dots, (z - z_i)^{k_{in}}) G(z)$$

where  $F(z)$  and  $G(z)$  are holomorphic and invertible. In the same trivialization, we can decompose  $\rho$  into its zeros and poles:

$$\rho_+ = F(z) \text{diag}((z - z_i)^{k_{i1}^+}, \dots, (z - z_i)^{k_{in}^+}) F^{-1}(z),$$

$$\rho_- = F(z) \text{diag}((z - z_i)^{k_{i1}^-}, \dots, (z - z_i)^{k_{in}^-}) G(z).$$

Denote by  $\rho(\mathcal{E}), \rho_+(\mathcal{E}), \rho_-(\mathcal{E})$  the Hecke modifications of  $\mathcal{E}$  corresponding to  $\rho, \rho_+, \rho_-$  respectively, as defined in [KW06]. Let  $m = \max_{i,j}(|k_{ij}|)$ .

Given  $\vec{t} = (t_1, \dots, t_N)$ , we consider the filtration

$$\mathcal{E}(mz_0) \supset \rho_-(\mathcal{E}) \supset \rho(\mathcal{E}) \supset \rho_+(\mathcal{E}) \supset \mathcal{E}(-mz_0),$$

we assign to it the following weights

$$(c_i, \min(c_i + t_i, 1 - c_i - t_i), \max(c_i + t_i, 1 - c_i - t_i), 1 - c_i),$$

here we choose positive real number  $c_i$  so that  $c_i < (1 - t_i)/2$ . In this way, we get a parabolic structure whose parabolic degree is

$$\deg \mathcal{E} - t_i \operatorname{Tr}(\vec{k}_i).$$

We repeat this process to each of the singularities and get a parabolic structure on  $\mathcal{E}(m(z_1 + \cdots + z_N))$ , corresponding to the divisor  $2m(z_1 + \cdots + z_N)$ . Its parabolic degree coincides with  $\vec{t}$ -degree. We can shift this parabolic structure to  $\mathcal{E}$ , though the parabolic degree changes, the stability conditions are equivalent.

In general, for a twisted bundle pair, one can still define (see e.g. [LM10]) twisted parabolic structure as in Definition 5.0.5, but with  $E_i$  being twisted bundles. We conjecture that there is a twisted parabolic structure whose stability condition corresponds to irreducible monopoles.

Note that the stability conditions on a bundle pair translate into a stability condition on its spectral datum. It is natural to conjecture that if the spectral cover is irreducible, we have a stable object. Explicitly, consider a spectral datum  $(\tilde{X}, \mathcal{L})$  and denote by  $cl(\tilde{X})$  the closure of  $\tilde{X}$  in  $X \times \mathbb{P}^1$ .

**Conjecture 5.0.6.** *If  $cl(\tilde{X})$  is reduced and irreducible, then  $\mathcal{L}$  extends to stable parabolic twisted sheaf on  $X \times \mathbb{P}^1$ , with pure support on  $cl(\tilde{X})$ , and it comes from an irreducible monopole on a circle bundle over  $X$ .*

*Remark 5.0.7.* Given a spectral datum  $(\tilde{X}, \mathcal{L})$ , we apply inverse Fourier-Mukai transform and get vector bundle  $E$  on a twisted  $S^1$  bundle over  $X$ . Furthermore, the

construction equips  $E$  with a canonical flat partial connection  $\nabla_E$ . Choose a unitary connection  $A$  on  $E$ , subtracting it from  $\nabla_E$  gives a Higgs field  $\phi = -\sqrt{-1}(\nabla_E - A_t)$ , so that  $(E, A, \phi)$  is triple that can be a candidate of monopole. Our conjecture says that when the spectral datum is stable, we can get a monopole in this way.

# Bibliography

- [AP01] D Arinkin and A Polishchuk. Fukaya category and Fourier transform. *AMS IP STUDIES IN ADVANCED MATHEMATICS*, 23:261–274, 2001.
- [BHM07] Dmitriy M Belov, Chris M Hull, and Ruben Minasian. T-duality, gerbes and loop spaces. *arXiv preprint arXiv:0710.5151*, 2007.
- [BMP01] Ugo Bruzzo, Giovanni Marelli, and Fabio Pioli. A Fourier transform for sheaves on real tori: Part (I). the equivalence  $\text{Sky}(T) \simeq \text{Loc}(\widehat{T})$ . *Journal of Geometry and Physics*, 39(2):174–182, 2001.
- [CH11] Benoit Charbonneau and Jacques Hurtubise. Singular Hermitian–Einstein monopoles on the product of a circle and a Riemann surface. *International Mathematics Research Notices*, 2011(1):175–216, 2011.
- [Dir31] Paul AM Dirac. Quantised singularities in the electromagnetic field. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 133(821):60–72, 1931.



- [DM96] Ron Donagi and Eyal Markman. Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles. In *Integrable systems and quantum groups*, pages 1–119. Springer, 1996.
- [DP03] Ron Donagi and Tony Pantev. Torus fibrations, gerbes, and duality. *arXiv preprint math/0306213*, 2003.
- [GG71] Jean Giraud and Jean Giraud. *Cohomologie non abélienne*, volume 179. Springer Berlin, 1971.
- [Hit82] Nigel J Hitchin. Monopoles and geodesics. *Communications in Mathematical Physics*, 83(4):579–602, 1982.
- [Hit01] Nigel Hitchin. Lectures on special Lagrangian submanifolds. *AMS IP Studies in Advanced Mathematics*, 23:151–182, 2001.
- [IM65] Nagayoshi Iwahori and Hideya Matsumoto. On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups. *Publications Mathématiques de l’IHÉS*, 25(1):5–48, 1965.
- [IS08] Jaya N Iyer and Carlos T Simpson. The Chern character of a parabolic bundle, and a parabolic corollary of Reznikovs theorem. In *Geometry and Dynamics of Groups and Spaces*, pages 439–485. Springer, 2008.
- [KW06] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *arXiv preprint hep-th/0604151*, 2006.

- [LM10] Marina Logares and Johan Martens. Moduli of parabolic Higgs bundles and Atiyah algebroids. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2010(649):89–116, 2010.
- [LT95] Martin Lübke and Andrei Teleman. *The Kobayashi-Hitchin correspondence*. World Scientific, 1995.
- [Moc04] Takuro Mochizuki. Kobayashi-Hitchin correspondence for tame harmonic bundles and an application. *arXiv preprint math/0411300*, 2004.